SPECTRA OF RANK-ONE PERTURBATIONS OF SELF-ADJOINT OPERATORS

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Abstract. We characterize possible spectra of rank-one perturbations B of a self-adjoint operator A with discrete spectrum and, in particular, prove that the spectrum of B may include any number of real or non-real eigenvalues of arbitrary algebraic multiplicity

1. Introduction

Assume that H is a separable Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and consider a self-adjoint operator A with simple discrete spectrum acting in H. Our aim is to study spectral properties of the rank-one perturbations of A, i.e., of the operators B of the form

$$B = A + \langle\cdot,\varphi\rangle\psi,$$

where φ and ψ are nonzero elements of H.

Rank-one perturbations of operators and matrices have been actively studied in both mathematical and physical literature for the reason that, on the one hand, they are simple enough to allow description of the spectral properties of perturbed operators via closed-form formulae which then can be analysed using various techniques; on the other hand, such perturbations turn out to be general enough to produce multitude of non-trivial effects.

One of the most general results in a finite-dimensional setting is given by Krupnik [21]; it states that rank-one perturbations of $n \times n$ matrices can produce arbitrary changes of eigenvalues. More exactly, assume $\lambda_1, \lambda_2, \dots, \lambda_p$ are pairwise distinct complex numbers and $k_1, k_2, \dots, k_p$ are natural numbers satisfying $k_1 + k_2 + \dots + k_p = n$; likewise, let $\mu_1, \mu_2, \dots, \mu_q$ be pairwise distinct complex numbers and $m_1, m_2, \dots, m_q$ be natural numbers satisfying $m_1 + m_2 + \dots + \mu_q = n$; then there is an $n \times n$ matrix A and its rank-one perturbation B such that $\lambda_1, \lambda_2, \dots, \lambda_p$

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are eigenvalues of A of respective multiplicities $k_1, k_2, \dots, k_p$, while the numbers $\mu_1, \mu_2, \dots, \mu_q$ are eigenvalues of B of respective multiplicities $m_1, m_2, \dots, m_q$. This statement is also specified to cases when both A and the perturbed matrix B belong to the Hermitian, unitary, or normal classes. Savchenko [33] studies the effect a generic rank-one perturbation has on the Jordan structure of a matrix A; an interesting observation is that, typically, in each root subspace, only the Jordan chain of the largest length splits; in [34], this is further generalized to low-rank perturbations, cf. also [31]. Similar results in infinite-dimensional Banach spaces were earlier derived by Hörmander and Melin in [19]. Bounds on the number of distinct eigenvalues of B in terms of some spectral characteristics of A are established in [16].

Structured perturbations of matrices and matrix pencils have recently been thoroughly studied in a series of papers by Mehl a.o. [25–30, 36]. Changes in the Jordan structures under perturbation within the classes of complex J-Hamiltonian and H-symmetric matrices and application in the control theory is discussed in [25]; see [27, 29] for further treatment in both the real and complex case. The class of H-Hermitian matrices, with (skew-)Hermitian H, is studied in [26, 28] via the canonical form of the pair (B, H). Rank-one perturbations of matrix pencils are discussed e.g. in [9, 17, 30]. A general perturbation theory for structured matrices is developed in the recent paper [36].

Typically, the above results exploit essentially matrix methods and thus are not directly applicable to the infinite-dimensional case (see, however, [19]). Rank-one perturbations of bounded or unbounded operators in infinite-dimensional Hilbert spaces have been studied within the general operator theory. For instance, a comprehensive spectral theory for rank-one perturbations of unbounded operators in the self-adjoint case is developed in [35], where a detailed characterization of discrete, absolutely continuous, and singlularly continuous components of the spectrum of the perturbed operator is given. A thorough overview of the theory of singular point perturbations of Schrödinger operators (formally corresponding to additive Dirac delta-functions and their derivatives) is given in the monographs by Albeverio a.o. [3, 7]. There has been much work devoted to the so-called singular and super-singular rank-one perturbations of self-adjoint operators, where the functions φ and ψ belong to the scales of Hilbert spaces $\operatorname{dom}(A^\alpha)$ with negative α, see e.g. [4–6, 8, 15, 18, 22–24]; in this case, a typical approach is through the Krein extension theory of self-adjoint operators. Rank-one and finite-rank perturbations of self-adjoint operators in Krein spaces have been recently discussed in [11, 12].

Despite the extensive research in the area, there seems to be no clear answer to the question, what are possible spectra of rank-one perturbations of a given operator in an infinite-dimensional space. The most pertinent work we are aware of includes the papers by Hörmader and Melin [19] and by Behrndt a.o. [10], which characterize possible changes in the Jordan structure of root subspaces of linear mappings in infinite-dimensional linear vector spaces under general finite-rank perturbations.

Our motivation in this work was to understand how the spectrum of a given operator in an infinite-dimensional Hilbert space can change under a rank-one perturbation, both locally, i.e., on the level of root subspaces, and globally, i.e., on the level of eigenvalue asymptotics. This task is quite non-trivial even in the case when the unperturbed operator A is self-adjoint but has generic spectrum, cf. [35]. Therefore, we decided to start with deriving a complete spectral picture in the simplest case where the unperturbed operator A is self-adjoint and has simple discrete spectrum; see Section 6 for discussion of possible generalizations. Under this assumption, our main result (Theorem 4.7) shows that the rank-one perturbation B of A may get eigenvalues of arbitrary algebraic multiplicities in an arbitrary finite set of points; however, all sufficiently large eigenvalues remain simple and asymptotically close to the eigenvalues of A. Note that a complete characterization of the possible spectra of rank-one perturbations of self-adjoint operators in Hilbert space, including precise asymptotic distribution and the constructive algorithm of finding φ and ψ, is suggested in a subsequent paper [14].

In the finite-dimensional case, our analysis leads to significant extensions of the result by Krupnik [21]; Theorem 5.1 states that if A is a given $n \times n$ Hermitian matrix with simple spectrum, then we can fix arbitrarily one of the vectors φ or ψ (say, φ) in some "generic" set so that, for any natural k, any pairwise distinct complex numbers $z_1, z_2, \dots, z_k$, and any natural numbers $m_1, m_2, \dots, m_k$ satisfying $m_1 + m_2 + \dots + m_k = n$, there is a unique ψ leading to a rank-one perturbed matrix B with eigenvalues $z_1, z_2, \dots, z_k$ of prescribed multiplicities $m_1, m_2, \dots, m_k$. We also specify this result in Theorem 5.2 to the case when φ or ψ is fixed arbitrarily.

The structure of the paper is as follows. In the next section, we introduce the characteristic function of the perturbed operator B and discuss how it is related to its spectrum. In Section 3, the algebraic multiplicities of eigenvalues are discussed, and in Section 4, the asymptotic distribution of eigenvalues is established. In Section 5, we specialize the obtained results to the finite-dimensional case, and in the final section we discuss possible generalizations of the main results to wider classes of the operators A.

2. General spectral properties of B

Throughout the paper, we make the following standing assumption on the operator A:

(A1) the operator A is self-adjoint and has simple discrete spectrum.

The operator A is necessarily unbounded; if A is semi-bounded, we shall assume that it is bounded below (otherwise, we shall study $-A$ instead). Therefore, under assumption (A1), the spectrum of A consists of real simple eigenvalues that can be listed in increasing order as λ_n, $n \in I$, with the index set I equal to $\mathbb{N}$ in the case where A is bounded below and to $\mathbb{Z}$ otherwise.

The operator B is a rank-one perturbation of the operator A, i.e.,

$$B = A + \langle \cdot, \varphi \rangle \psi \tag{2.1}$$

with fixed nonzero vectors φ and ψ in H. Clearly, the operator B is well defined and closed on its natural domain $\operatorname{dom}(B)$ equal to $\operatorname{dom}(A)$. Next, for λ in the resolvent set $\rho(A)$ of A, we introduce the *characteristic function*

$$F(\lambda) := \langle (A-\lambda)^{-1}\psi, \varphi \rangle + 1 \tag{2.2}$$

and denote by $\mathcal{N}_F$ the set of zeros of F. Many spectral properties of the operator B of (2.1) will be derived from the explicit formula for its resolvent known as the Krein formula (see, e.g., [7, Sec. 1.1.1]); we include its proof for the sake of completeness and to derive some explicit relations to be used later on.

Lemma 2.1 (The Krein formula). *The set $\rho(A) \setminus \mathcal{N}_F$ consists of resolvent points of the operator B and, for every $\lambda \in \rho(A) \setminus \mathcal{N}_F$,*

$$(B-\lambda)^{-1} = (A-\lambda)^{-1} - \frac{\langle \cdot, (A-\overline{\lambda})^{-1}\varphi \rangle}{F(\lambda)} (A-\lambda)^{-1}\psi. \tag{2.3}$$

Proof. To prove that a fixed $\lambda \in \rho(A) \setminus \mathcal{N}_F$ is a resolvent point of B, we need to show that for every $g \in H$ the equation

$$g = (B-\lambda)f \tag{2.4}$$

can be uniquely solved for $f \in H$. Assuming such an f exists, writing the equality (2.4) as

$$g = (A-\lambda)f + \langle f, \varphi \rangle \psi, \tag{2.5}$$

and applying the resolvent of the operator A to both sides, we obtain

$$(A-\lambda)^{-1}g = f + \langle f, \varphi \rangle (A-\lambda)^{-1}\psi. \tag{2.6}$$

Taking the inner product with φ results in the equality

$$\langle (A-\lambda)^{-1}g,\varphi\rangle = \langle f,\varphi\rangle + \langle f,\varphi\rangle\langle (A-\lambda)^{-1}\psi,\varphi\rangle = \langle f,\varphi\rangle F(\lambda), \tag{2.7}$$

which on account of $F(\lambda)\neq 0$ leads to

$$\langle f,\varphi\rangle = \frac{\langle (A-\lambda)^{-1}g,\varphi\rangle}{F(\lambda)}. \tag{2.8}$$

Substituting now (2.8) in (2.6), we derive the following formula for f:

$$f = (A-\lambda)^{-1}g - \frac{\langle (A-\lambda)^{-1}g,\varphi\rangle}{F(\lambda)}(A-\lambda)^{-1}\psi. \tag{2.9}$$

A direct verification shows that f of (2.9) belongs to $\operatorname{dom}(B)=\operatorname{dom}(A)$ and is indeed a solution of equation (2.4).

Therefore, the operator $B-\lambda$ is surjective. It is also injective since if an $f\in\operatorname{dom}(B)$ satisfies (2.4) with $g=0$, then (2.6) on account of (2.8) gives $f=0$. Thus the operator $B-\lambda$ is invertible and its inverse is equal to

$$(B-\lambda)^{-1} = (A-\lambda)^{-1} - \frac{\langle\,\cdot\,,(A-\overline{\lambda})^{-1}\varphi\rangle}{F(\lambda)}\,(A-\lambda)^{-1}\psi$$

as claimed. The proof is complete. □

The Krein formula shows that, for every $\lambda\in\rho(A)\setminus\mathcal{N}_F$, the resolvent $(B-\lambda)^{-1}$ is a rank-one perturbation of the compact operator $(A-\lambda)^{-1}$. Therefore, we get the following

Corollary 2.2. *The resolvent of the operator B is compact, i.e., B is an operator with discrete spectrum.*

Next we denote by v_n a normalized eigenvector of A corresponding to its eigenvalue λ_n; then the set $\{v_n\}_{n\in I}$ is an orthonormal basis of H. We also denote by a_n and b_n the Fourier coefficients of the vectors φ and ψ with respect to this basis, so that[1]

$$\varphi = \sum_{n\in I} a_n v_n, \qquad \psi = \sum_{n\in I} b_n v_n.$$

Lemma 2.3. *The following relations hold between the spectra of the operators A and B:*

a) *for $\lambda\in\rho(A)$, λ belongs to the spectrum of B if and only if $\lambda\in\mathcal{N}_F$;*

b) *the eigenvalue $\lambda=\lambda_n$ of the operator A belongs to spectrum of the operator B if and only if $a_nb_n=0$.*

[1] In the case $I=\mathbb{Z}$, the summation will always be understood in the principal value sense

Proof. a) Let a point $\lambda \in \rho(A)$ belong to the spectrum of the operator B. By Corollary 2.2, λ is an eigenvalue of the operator B, and we denote by y a corresponding eigenvector. Then (2.4) holds with $g = 0$ and with y in place of f, so that equations (2.6) and (2.7) can be recast as

$$y = -\langle y, \varphi\rangle (A - \lambda)^{-1}\psi$$

and

$$\langle y, \varphi\rangle F(\lambda) = 0,$$

respectively. Since y is a nonzero vector, we see from the former equality that $\langle y, \varphi\rangle \neq 0$, and then the latter one yields $F(\lambda) = 0$.

Conversely, if $F(\lambda) = 0$ for some $\lambda \in \rho(A)$, then $y := (A - \lambda)^{-1}\psi$ is an eigenvector of the operator B for the eigenvalue λ, as is seen from the equalities

$$(A - \lambda)y + \langle y, \varphi\rangle\psi = [1 + \langle (A - \lambda)^{-1}\psi, \varphi\rangle]\psi = F(\lambda)\psi = 0.$$

This completes the proof of part a).

b) Let $\lambda = \lambda_n$ belong to the spectrum of the operator B; then there is a vector $y \in \operatorname{dom}(B)$ such that $By = \lambda_n y$, i.e.,

$$(B - \lambda_n)y = (A - \lambda_n)y + \langle y, \varphi\rangle\psi = 0. \tag{2.10}$$

Taking the inner product with v_n results in

$$\begin{aligned}\langle (A - \lambda_n)y, v_n\rangle + \langle y, \varphi\rangle\langle \psi, v_n\rangle &= \langle y, (A - \lambda_n)v_n\rangle + \langle y, \varphi\rangle\langle \psi, v_n\rangle \\ &= \langle y, \varphi\rangle\langle \psi, v_n\rangle = 0.\end{aligned}$$

Thus $\langle y, \varphi\rangle = 0$ or $\langle \psi, v_n\rangle = 0$. If $\langle y, \varphi\rangle = 0$, then $y = cv_n$ for some constant c on account of (2.10), so that $a_n = 0$. If $\langle \psi, v_n\rangle = 0$, then $b_n = 0$. Therefore the point $\lambda = \lambda_n$ belongs to the spectrum of B only if $a_n b_n = 0$.

Conversely, let $a_n b_n = 0$; we need to prove that the point $\lambda = \lambda_n$ belongs to the spectrum of B. If $a_n = 0$, then

$$(B - \lambda_n)v_n = (A - \lambda_n)v_n + \overline{a_n}\psi = 0$$

so that $y = v_n$ is an eigenvector of B for the eigenvalue λ_n. If $b_n = 0$, then for all $y \in \operatorname{dom}(B)$

$$\langle (B - \lambda_n)y, v_n\rangle = \langle (A - \lambda_n)y, v_n\rangle = \langle y, (A - \lambda_n)v_n\rangle = 0,$$

so that $B - \lambda_n$ is not surjective on $\operatorname{dom}(B)$ and the point $\lambda = \lambda_n$ belongs to the spectrum of the operator B. The proof is complete. □

We introduce the sets of indices

$$I_0 \stackrel{\text{def}}{=} \{n \in I \mid a_n b_n = 0\}, \quad I_1 \stackrel{\text{def}}{=} \{n \in I \mid a_n b_n \neq 0\}$$

of (possibly infinite) cardinalities N_0 and N_1 respectively, and split the eigenvalues of A into the respective subsets

$$\sigma_0(A) \stackrel{\text{def}}{=} \{\lambda_n \,|\, n \in I_0\} \qquad \text{and} \qquad \sigma_1(A) \stackrel{\text{def}}{=} \{\lambda_n \,|\, n \in I_1\}.$$

According to Lemma 2.3, the spectrum of the operator B consists of two parts: $\sigma_0(A) = \sigma(A) \cap \sigma(B)$, the common eigenvalues of A and B, and the set $\mathcal{N}_F$ of zeros of the function F in $\rho(A)$. Certainly, the latter part of $\sigma(B)$ is more interesting.

3. Eigenvalue multiplicity

In this section, we discuss multiplicity of eigenvalues of the operator B.

First we recall that the *geometric multiplicity* of an eigenvalue λ of an operator T is the dimension of the corresponding eigenspace, i.e., the number $\dim\ker(T-\lambda)$ [20, Ch. 5.1], and its *algebraic multiplicity* is the dimension of the corresponding root subspace, i.e., the rank of the corresponding spectral projector [20, Ch. 5.4]. Note that for a selfadjoint operator geometric and algebraic multiplicities of every eigenvalue are equal.

Before proceeding, we recall that the function F was initially defined only on the resolvent set of the operator A. However, using the spectral theorem for the operator A, we can write the function F as

$$F(\lambda) = \sum_{n\in I_1} \frac{\overline{a_n} b_n}{\lambda_n - \lambda} + 1, \tag{3.1}$$

and this formula gives an analytic continuation of F onto the set $\sigma_0(A)$; we shall denote this continuation by the same letter F.

Lemma 3.1. *An eigenvalue λ of B has geometric multiplicity larger than 1 if and only if there exists an integer n such that $\lambda = \lambda_n$, $a_n = b_n = 0$, and $F(\lambda_n) = 0$. In that case, the geometric multiplicity of λ is equal to 2.*

Proof. Assume that $\lambda \in \sigma(B)$ has geometric multiplicity larger than 1, and denote by y any of the corresponding eigenvectors. Then

$$(B-\lambda)y = (A-\lambda)y + \langle y, \varphi\rangle\psi = 0,$$

and if λ is a resolvent point of A, then y must be collinear to the vector $(A-\lambda)^{-1}\psi$ and thus the geometric multiplicity of λ is one. Therefore, $\lambda \in \sigma(A)$, so that $\lambda = \lambda_n$ for some $n \in I_0$. Now, as in the proof of part b) of Lemma 2.3, we find that

$$0 = \langle (B-\lambda_n)y, v_n\rangle = \langle y, \varphi\rangle\langle\psi, v_n\rangle = \langle y, \varphi\rangle b_n,$$

so that $\langle y, \varphi\rangle = 0$ or $b_n = 0$.

Assume that $b_n \neq 0$; then $\langle y, \varphi\rangle = 0$ and $(B - \lambda_n)y = (A - \lambda_n)y = 0$. Thus y in that case must be collinear to v_n, and the geometric multiplicity of λ_n is then 1. Therefore, $b_n = 0$ and the vector ψ belongs to the subspace $H_n := H \ominus \langle v_n\rangle$. Since the nullspace of $B - \lambda_n$ is of dimension at least 2, there is an eigenvector w in H_n. We denote by A_n the restriction $A|_{H_n}$ of A onto its invariant subspace H_n and see that

$$(A_n - \lambda_n)w + \langle w, \varphi\rangle\psi = 0.$$

Note that λ_n is a resolvent point of the operator A_n, so that the above equality implies that $w = c(A_n - \lambda_n)^{-1}\psi$ and that

$$\langle (A_n - \lambda_n)^{-1}\psi, \phi\rangle + 1 = 0,$$

i.e., that $F(\lambda_n) = 0$. Therefore, there is at most one (up to a factor) eigenvector of B in the space H_n, and thus its second eigenvector must be of the form $v_n + w_n$ with some $w_n \in H_n$. However, then

$$(B - \lambda_n)(v_n + w_n) = (A - \lambda_n)w_n + \langle v_n + w_n, \varphi\rangle\psi = 0$$

so that w_n is collinear to the eigenvector $(A_n - \lambda_n)^{-1}\psi$ found earlier, and thus v_n must also be an eigenvector of B. As $(B-\lambda_n)v_n = \langle v_n, \varphi\rangle\psi$, this requires that $a_n = 0$.

Summing up, we see that the assumption that $\dim\ker(B - \lambda) > 1$ implies that $\lambda = \lambda_n$ for some $n \in I$ and $b_n = 0$; moreover, there is an eigenvector w in the subspace H_n if and only if $F(\lambda_n) = 0$, and then w is collinear to $(A_n - \lambda_n)^{-1}\psi$. The second eigenvector must be v_n, which is possible if and only if $a_n = 0$. Therefore all the conditions are necessary, and the geometric multiplicity is then equal to 2.

To prove that these conditions are also sufficient, we assume that $\lambda = \lambda_n$ is such that $a_n = b_n = 0$ and $F(\lambda_n) = 0$. Then, as shown above, v_n and $w := (A_n - \lambda_n)^{-1}\psi \in H_n$ are linearly independent eigenvectors of B for the eigenvalue λ_n. The proof is complete. □

Example 3.2. *Let λ and μ be distinct eigenvalues of an operator A with corresponding normalized eigenvectors v and w; then for the operator $B := A+(\lambda-\mu)\langle\cdot, w\rangle w$ the number λ is an eigenvalue of geometric multiplicity two, v and w being the corresponding eigenvectors. As the above lemma shows, geometric multiplicity cannot be made larger by a rank-one perturbation of A.*

Remark 3.3. *Assume that $a_n = b_n = 0$, so that λ_n is an eigenvalue of B with eigenvector v_n. Then v_n is also an eigenvector of the adjoint operator B^*, so that the subspaces $\langle v_n\rangle$ and H_n are reducing for B. Moreover, the restrictions of A and B onto $\langle v_n\rangle$ coincide.*

More generally, we denote by H^0 *the closed linear space of all eigenvectors* v_k *of* A *for which* $a_k = b_k = 0$*. Then the subspace* H^0 *is reducing for* B *and the restrictions of* A *and of* B *onto* H^0 *coincide. Therefore, we can concentrate on the study of the restriction of the operator* B *onto its invariant subspace* $H^1 := H \ominus H^0$*. Without loss of generality, we shall assume that* $H^0 = \{0\}$*, so that* $H = H^1$*. Under this assumption, all eigenvalues of the operator* B *are geometrically simple.*

Next we discuss algebraic multiplicity of the eigenvalues of B in the resolvent set of A. As every such an eigenvalue λ is geometrically simple by Lemma 3.1, its algebraic multiplicity coincides with the largest length of chains of eigen- and associated vectors (also called Jordan chains). We recall that a sequence of vectors $y_0, y_1, \dots, y_m$ forms a chain of eigen- and associated vectors of B for an eigenvalue λ if every y_k is in the domain of B, $(B-\lambda)y_0 = 0$, and $(B-\lambda)y_k = y_{k-1}$ for $k = 1, \dots, m$. Chains of eigen- and associated vectors are not defined uniquely; however, for geometrically simple eigenvalues all such chains are closely related, as the next lemma demonstrates.

Lemma 3.4. *Assume that* λ *is a (geometrically simple) eigenvalue of the operator* B *and* $y_0, y_1, \dots, y_m$ *is a chain of eigen- and associated vectors corresponding to* λ*.*

(i) *For every sequence of complex numbers* $c_1, \dots, c_m$ *introduce the vectors* $\tilde{y}_0 = y_0$ *and*

$$\tilde{y}_k = y_k + c_1 y_{k-1} + \dots + c_k y_0 \tag{3.2}$$

for $k = 1, \dots, m$*. Then* $\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_m$ *is a chain of eigen- and associated vectors of* B *corresponding to* λ*.*

(ii) *Vice versa, assume that* $\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_m$ *is another chain of eigen- and associated vectors of* B *corresponding to the eigenvalue* λ *such that* $\tilde{y}_0 = y_0$*. Then there are constants* $c_1, \dots, c_m$ *such that for all* $k = 1, 2, \dots, m$ *relations* (3.2) *hold.*

Proof. By definition of $\tilde{y}_k$ and y_k, we find that

$$(B-\lambda)\tilde{y}_k = y_{k-1} + c_1 y_{k-1} + \dots + c_{k-1} y_0 = \tilde{y}_{k-1}$$

for $k \ge 1$ thus establishing Part (i).

For part (ii), the proof is by induction. Since $(B-\lambda)(\tilde{y}_1 - y_1) = \tilde{y}_0 - y_0 = 0$, it follows that there is $c_1 \in \mathbb{C}$ such that $\tilde{y}_1 - y_1 = c_1 y_0$ thus establishing the base of induction. Assume that the claim has already been proved for $k = 1, \dots, l-1 < m$; then

$$(B-\lambda)(\tilde{y}_l - y_l) = \tilde{y}_{l-1} - y_{l-1} = c_1 y_{l-2} + \dots + c_{l-1} y_0$$

and

$$(B-\lambda)(\tilde{y}_l - y_l - c_1 y_{l-1} - \cdots - c_{l-1} y_1) = 0.$$

Therefore, there exists a number $c_l \in \mathbb{C}$ such that $\tilde{y}_l - y_l - c_1 y_{l-1} - \cdots - c_{l-1} y_1 = c_l y_0$, thus finishing the induction step and the proof of the lemma. ☐

Lemma 3.5. *Let $\lambda^* \in \rho(A)$ be an eigenvalue of the operator B. Then its algebraic multiplicity coincides with the multiplicity of λ^* as a zero of the characteristic function F.*

Proof. By Lemma 2.3, $F(\lambda^*) = 0$, and the proof of that lemma shows that the vector $y_0 := (A-\lambda^*)^{-1}\psi$ is an eigenvector of B for the eigenvalue λ^*. Denote by $l+1$ the multiplicity of zero $\lambda = \lambda^*$ of F and set $y_k := (A-\lambda^*)^{-(1+k)}\psi$ for $k = 1, 2, \dots, l$. Recall [20, §III.6] that the resolvent $(A-\lambda)^{-1}$ is differentiable on the set $\rho(A)$ and that

$$\frac{d}{d\lambda}(A-\lambda)^{-1} = (A-\lambda)^{-2}.$$

Observing now that

$$\frac{1}{k!}F^{(k)}(\lambda^*) = \langle (A-\lambda^*)^{-(1+k)}\psi, \varphi\rangle = \langle y_k, \varphi\rangle \tag{3.3}$$

for $k \in \mathbb{N}$, we find that

$$(B-\lambda^*)y_k = (A-\lambda^*)y_k + \langle y_k, \varphi\rangle\psi = y_{k-1} + \frac{1}{k!}F^{(k)}(\lambda^*)\psi = y_{k-1}$$

for $k = 1, \dots, l$. Thus $y_0, y_1, \dots, y_l$ is a Jordan chain of the operator B for the eigenvalue λ^*, so that the algebraic multiplicity of λ^* is at least $l+1$.

Assume that $\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_m$ is a Jordan chain of B for the eigenvalue λ^*. Then $\tilde{y}_0$ is an eigenvector of B, and without loss of generality we can assume that $\tilde{y}_0 = y_0$. Now we prove by induction that, with $c_k := -\langle \tilde{y}_k, \varphi\rangle$ for $k = 1, \dots, m$, we have

$$\tilde{y}_k = y_k + c_1 y_{k-1} + \cdots + c_k y_0 \tag{3.4}$$

and that $F(\lambda^*) = F'(\lambda^*) = \cdots = F^{(m)}(\lambda^*) = 0$.

Indeed, $F(\lambda^*) = 0$ by Lemma 2.3, and the relation

$$(B-\lambda^*)\tilde{y}_1 = (A-\lambda^*)\tilde{y}_1 + \langle \tilde{y}_1, \varphi\rangle\psi = y_0$$

shows that $\tilde{y}_1 = y_1 + c_1 y_0$. Taking now the scalar product with φ and recalling that $\langle y_0, \varphi\rangle = -1$, we get the equality $\langle y_1, \varphi\rangle = 0$ resulting in $F'(\lambda^*) = 0$ in view of (3.3) and establishing the base of induction.

Assuming that the relations $F(\lambda^*) = F'(\lambda^*) = \cdots = F^{(k)}(\lambda^*) = 0$ and $\tilde{y}_k = y_k + c_1 y_{k-1} + \cdots + c_k y_0$ have already been proved for some $k < m$, we recast the equality $(B - \lambda^*)\tilde{y}_{k+1} = \tilde{y}_k$ as

$$(A - \lambda^*)\tilde{y}_{k+1} + \langle \tilde{y}_{k+1}, \varphi\rangle\psi = y_k + c_1 y_{k-1} + \cdots + c_k y_0.$$

Applying $(A - \lambda^*)^{-1}$ to both sides of the above equality leads to the relation

$$\tilde{y}_{k+1} = y_{k+1} + c_1 y_k + \cdots + c_k y_1 + c_{k+1} y_0,$$

which on account of (3.3) and the induction assumption yields

$$\begin{aligned} c_{k+1} &:= -\langle \tilde{y}_{k+1}, \varphi\rangle = -\langle y_{k+1} + c_1 y_k + \cdots + c_k y_1 + c_{k+1} y_0, \varphi\rangle \\ &= -\frac{1}{(k+1)!} F^{(k+1)}(\lambda^*) + c_{k+1} \end{aligned}$$

and $F^{(k+1)}(\lambda^*) = 0$, thus completing the induction step. Therefore, λ^* is a zero of F of multiplicity at least $m+1$, and the proof is complete. □

Example 3.6. *To demonstrate the above result, we show how to "move" any $m+1$ eigenvalues of a generic operator A satisfying (A1) into an arbitrary point $\lambda \notin \sigma(A)$. For definiteness, we choose the eigenvalues $\lambda_0, \dots, \lambda_m$ and a point $\lambda = i$. We shall take both ϕ and ψ in the linear span of the eigenvectors $v_0, v_1, \dots, v_m$, i.e.,*

$$\phi = \sum_{n=0}^{m} a_n v_n, \qquad \psi = \sum_{n=0}^{m} b_n v_n$$

and set $c_n := \overline{a_n} b_n$ for $n = 0, 1, \dots, m$. The corresponding characteristic function can be then written as

$$F(\lambda) = \sum_{n=0}^{m} \frac{c_n}{\lambda_n - \lambda} + 1 = \frac{p(\lambda)}{\prod_{n=0}^{m}(\lambda - \lambda_n)},$$

with a monic polynomial p of degree $m+1$. According to the above lemma, F must satisfy the equalities $F(i) = F'(i) = \cdots = F^{(m)}(i) = 0$, which implies that $p(\lambda) = (\lambda - i)^{m+1}$. Then we find that

$$c_n = -\operatorname*{res}_{\lambda=\lambda_n} F(\lambda) = -\frac{(\lambda_n - i)^{m+1}}{\prod_{k\neq n}(\lambda_n - \lambda_k)}$$

and can choose, e.g., $a_n = \overline{c_n}$ and $b_n = 1$ for $n = 0, 1, \dots, m$.

In particular, $\phi = \sum_{n=0}^{m} \overline{c_n} v_n$, $\psi = \sum_{n=0}^{m} v_n$, and the vectors

$$y_k := (A - i)^{k+1}\psi = \sum_{n=0}^{m} \frac{v_n}{(\lambda_n - i)^{k+1}}, \qquad k = 0, 1, \dots, m,$$

satisfy the relations

$$\begin{aligned}(B-i)y_k &= (A-i)y_k + \langle y_k, \phi\rangle\psi \\ &= \sum_{n=0}^{m} \frac{v_n}{(\lambda_n - i)^k} + \sum_{n=0}^{m} \frac{c_n}{(\lambda_n - i)^{k+1}}\psi \\ &= \begin{cases} F(i)\psi, & k=0 \\ y_{k-1} + k!F^{(k)}(i)\psi, & k=1,\dots,m. \end{cases}\end{aligned}$$

In view of the relations $F(i) = F'(i) = \dots = F^{(m)}(i) = 0$*, these vectors form a chain of eigen- and associated vectors of* B *for the eigenvalue* i*.*

As we noted in Lemma 2.3, every point $\lambda = \lambda_n$ of $\sigma_0(A)$ is also an eigenvalue of B. We agreed earlier to exclude the non-interesting case where $a_n = b_n = 0$, which by Lemma 3.1 means that such a λ_n is a geometrically simple eigenvalue of B. However, its algebraic multiplicity may be greater than one, and we shall relate it to the multiplicity of λ_n as a zero of the function F; recall that F was extended by continuity to the set $\sigma_0(A)$ by formula (3.1).

Lemma 3.7. *Assume that* $\lambda_n \in \sigma_0(A)$ *is an eigenvalue of* B *of geometric multiplicity* 1 *and algebraic multiplicity* $m \geq 1$*. Denote by* l *multiplicity of* λ_n *as a zero of the function* F*; then* $m = l+1$*.*

Proof. Denote by y_0 an eigenfunction of B corresponding to the eigenvalue λ_n. Since by assumption λ_n is a geometrically simple eigenvalue, y_0 is determined uniquely up to a constant factor. As was shown in the proof of Lemma 2.3,

$$0 = \langle (B-\lambda_n)y_0, v_n\rangle = \langle y_0, \varphi\rangle\langle \psi, v_n\rangle = \langle y_0, \varphi\rangle b_n$$

so that either $\langle y_0, \varphi\rangle = 0$ or $b_n = 0$. We shall analyse these two cases separately.

Case (a): $\langle y_0, \varphi\rangle = 0$. Then $(B-\lambda_n)y_0 = (A-\lambda_n)y_0 = 0$, so that y_0 can be taken equal to v_n. As a result, $\overline{a_n} = \langle v_n, \varphi\rangle = 0$, i.e., $\varphi \in H_n := H \ominus v_n$.

If a vector y_1 is associated to the eigenvector y_0, then y_1 should satisfy the relation

$$(B-\lambda_n)y_1 = (A-\lambda_n)y_1 + \langle y_1, \varphi\rangle\psi = y_0 = v_n \tag{3.5}$$

and is determined up to the eigenvector $y_0 = v_n$. Therefore, if such a vector y_1 exists, it can be chosen orthogonal to v_n, i.e., from H_n. Then after taking the scalar product of $(B-\lambda_n)y_1$ with v_n and recalling that $\operatorname{ran}(A-\lambda_n) = H_n$, we conclude from (3.5) that

$$\langle y_1, \varphi\rangle\langle \psi, v_n\rangle = \langle y_1, \varphi\rangle b_n = 1.$$

Thus b_n must be nonzero and $\langle y_1, \varphi\rangle = 1/b_n$.

Denote by P_n the orthogonal projector onto the subspace H_n and by A_n the restriction of the operator A onto the subspace H_n; then λ_n is a resolvent point of A_n. Applying P_n to (3.5), we conclude that

$$(A_n - \lambda_n)y_1 + \frac{1}{b_n}P_n\psi = 0,$$

so that $y_1 = -\frac{1}{b_n}(A_n - \lambda_n)^{-1}P_n\psi$. The norming condition $\langle y_1, \varphi\rangle = 1/b_n$ can now be recast as

$$\langle (A_n - \lambda_n)^{-1}P_n\psi, \varphi\rangle + 1 = \langle \psi, (A_n - \lambda_n)^{-1}\varphi\rangle + 1 = 0$$

and amounts to the equality $F(\lambda_n) = 0$. The conclusion is that an associated vector y_1 exists if and only if $b_n \neq 0$ and λ_n is a zero of F (i.e. $l > 0$). In particular, $l = 0$ is equivalent to $m = 1$ (recall that the case $a_n = b_n = F(\lambda_n) = 0$ was excluded), and the equality $m = l + 1$ is then satisfied.

Assume therefore that $l > 0$ and introduce the vectors

$$y_k := -\frac{1}{b_n}(A_n - \lambda_n)^{-k}P_n\psi, \qquad k \geq 1.$$

Then one sees that

$$(B - \lambda_n)y_k = (A - \lambda_n)y_k + \langle y_k, \varphi\rangle\psi = y_{k-1} + \langle y_k, \varphi\rangle\psi$$

and

$$\langle y_k, \varphi\rangle = -\frac{1}{b_n}\langle (A_n - \lambda_n)^{-k}P_n\psi, \varphi\rangle = -\frac{1}{b_n(k-1)!}F^{(k-1)}(\lambda_n).$$

It follows that the vectors $y_1, y_2, \dots, y_l$ form a chain of vectors associated to the eigenvector y_0, so that the algebraic multiplicity m of the eigenvalue λ_n is at least $l + 1$.

Conversely, as in the proof of Lemma 3.5 one can show that in any chain $\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_{m-1}$ of eigen- and associated vectors for B the vectors $\tilde{y}_1, \dots, \tilde{y}_{m-1}$ are related to the above-constructed vectors $y_1, \dots, y_{m-1}$ via (3.4) and that $F(\lambda_n) = F'(\lambda_n) = \dots = F^{(m-2)}(\lambda_n) = 0$. This shows that $l \geq m - 1$ and completes the proof in the case (a).

Case (b): $b_n = 0$. Then ψ belongs to $H_n = H \ominus v_n$ and thus the range $\operatorname{ran}(B - \lambda_n)$ of $B - \lambda_n$ is contained in H_n. We look for an eigenvector y_0 of B of the form $\alpha_0 v_n + z_0$ with $z_0 \in H_n$. Then $(A - \lambda_n)y_0 = (A_n - \lambda_n)z_0$, and $(B - \lambda_n)y_0 = 0$ can be written as

$$(A_n - \lambda_n)z_0 + \langle y_0, \varphi\rangle\psi = 0,$$

so that $z_0 = c(A_n - \lambda_n)^{-1}\psi$ with an appropriate constant c. Substituting this z_0 into the above equation results in the relation

$$c\psi + \big[\alpha_0\overline{a_n} + c\langle (A_n - \lambda_n)^{-1}\psi, \varphi\rangle\big]\psi = 0,$$

yielding the equality

$$cF(\lambda_n) + \alpha_0 \overline{a_n} = 0. \tag{3.6}$$

In order that for the eigenvector y_0 there could exist an associated vector y_1, it is necessary that $y_0 = (B - \lambda_n)y_1$ belong to H_n and thus that $\alpha_0 = 0$ and $y_0 = z_0$. Equation (3.6) then yields $cF(\lambda_n) = 0$, and since $c = 0$ would lead to the contradiction that $y_0 = z_0 = 0$, we conclude that necessarily $F(\lambda_n) = 0$. In particular, $l = 0$ gives $m = 1$ as stated.

Assume therefore that $l > 0$, so that $F(\lambda_n) = 0$. As the case $a_n = b_n = 0$ was excluded earlier, we have $a_n \neq 0$ and thus $\alpha_0 = 0$ by (3.6) and $y_0 = z_0 := (A_n - \lambda_n)^{-1}\psi$.

We first show that $m \geq l + 1$ by constructing a chain $y_1, \dots, y_l$ of vectors associated to this y_0. Namely, take $y_k := (A_n - \lambda_n)^{-(1+k)}\psi$ for $k = 1, \dots, l - 1$ and $y_l := \alpha_l v_n + (A_n - \lambda_n)^{-(1+l)}\psi$ with an α_l to be determined later. As in the proof of Case (a) we find that

$$(B - \lambda_n)y_k = (A_n - \lambda_n)y_k + \langle y_k, \varphi\rangle\psi = y_{k-1} + \frac{1}{k!}F^{(k)}(\lambda_n)\psi = y_{k-1}$$

for $k = 1, 2, \dots, l - 1$. For $k = l$ we get

$$(B - \lambda_n)y_l = (A_n - \lambda_n)y_l + \langle y_l, \varphi\rangle\psi = y_{l-1} + \big[\alpha_l \overline{a_n} + \frac{1}{l!}F^{(l)}(\lambda_n)\big]\psi,$$

and the equality $(B - \lambda_n)y_l = y_{l-1}$ is guaranteed by taking (recall that $a_n \neq 0$)

$$\alpha_l := -\frac{1}{\overline{a_n} l!}F^{(l)}(\lambda_n).$$

It remains to show that $l \geq m - 1$. We take a chain of eigen- and associated vectors $\tilde{y}_0, \dots, \tilde{y}_{m-1}$ of the maximal possible length $m > 1$. The equalities $(B - \lambda_n)\tilde{y}_k = \tilde{y}_{k-1}$ for $k = 1, \dots, m - 1$ show that the vectors $\tilde{y}_0, \dots, \tilde{y}_{m-2}$ belong to H_n. Without loss of generality we may assume that $\tilde{y}_0 = y_0$ and then prove by induction that with $c_k := -\langle \tilde{y}_k, \varphi\rangle$ for $k = 0, 1, \dots, m - 2$ we have

$$\tilde{y}_k = y_k + c_1 y_{k-1} + \dots + c_k y_0$$

with y_k defined above and that $F(\lambda_n) = F'(\lambda_n) = \dots = F^{(k)}(\lambda_n) = 0$.

The base of induction was already set up: $\tilde{y}_0 = y_0$ and $F(\lambda_n) = 0$. Assume therefore that the claim holds for all indices k less than j with $0 < j < m - 2$ and rewrite the equality $(B - \lambda_n)\tilde{y}_j = \tilde{y}_{j-1}$ as

$$(A_n - \lambda_n)\tilde{y}_j + \langle \tilde{y}_j, \varphi\rangle\psi = \tilde{y}_{j-1}.$$

It follows that $\tilde{y}_j = (A_n - \lambda_n)^{-1}\tilde{y}_{j-1} - \langle \tilde{y}_j, \varphi\rangle y_0$, which by the induction assumption can be recast as

$$\tilde{y}_j = y_j + c_1 y_{j-1} + \cdots + c_j y_0.$$

Since

$$\frac{1}{k!}F^{(k)}(\lambda_n) = \langle y_k, \varphi\rangle$$

for $k \in \mathbb{N}$ and the equalities $F(\lambda_n) = F'(\lambda_n) = \cdots = F^{(j-1)}(\lambda_n) = 0$ hold by assumption, we find that $\langle y_0, \varphi\rangle = -1$ and, by taking the scalar product with φ in the above formula for $\tilde{y}_j$, that

$$-c_j = \langle \tilde{y}_j, \varphi\rangle = \langle y_j, \varphi\rangle - c_j.$$

Therefore $\langle y_j, \varphi\rangle = 0$ yielding the relation $F^{(j)}(\lambda_n) = 0$.

This completes the induction step and shows that λ_n is a zero of F of multiplicity at least $m-1$. The proof is complete. □

Example 3.8. *In the Hilbert space $L_2(0, 2\pi)$, we consider a self-adjoint operator*

$$A = \frac{1}{i}\frac{d}{dx}$$

subject to the periodic boundary condition $y(0) = y(2\pi)$. The spectrum of A coincides with the set $\mathbb{Z}$, and an eigenfunction v_n corresponding to the eigenvalue $\lambda_n := n$ is equal to $e^{inx}/\sqrt{2\pi}$.

For every $m \in \mathbb{N}$, we shall construct a rank-one perturbation $\langle\,\cdot\,, \phi\rangle\psi$ so that the perturbed operator B has an eigenvalue $\lambda_0 = 0$ of algebraic multiplicity $2m+1$ and simple eigenvalues $\mu_n = \lambda_n$ if $|n| > m$. More precisely, we take

$$\phi(x) = \sum_{k=-m}^{m} e^{ikx} = \frac{\sin(m+\frac{1}{2})x}{\sin(\frac{1}{2}x)}$$

and

$$\psi(x) = \sum_{k=1}^{m} d_k \sin(kx)$$

with coefficients d_k to be determined. Since $\langle \psi, v_0\rangle = 0$, the corresponding chain of eigen- and associated vectors can be formed as in Case (b) of the above theorem. Namely, with A_0 standing for the restriction of A onto the space $H_0 := H \ominus v_0$, we take

$$y_k := A_0^{-(k+1)}\psi, \qquad k = 0, \dots, 2m-1,$$

and

$$y_{2m} := d_0 v_0 + A_0^{-(2m+1)}\psi$$

for a suitable d_0. We next show that there is a unique set of $d_0, \dots, d_m$ for which the above $y_0, \dots, y_{2m}$ form a chain of eigen- and associated vectors of B corresponding to λ_0 and that there is no longer chains.

Note that

$$A_0^{-2l}\psi(x) = \sum_{k=1}^{m} \frac{d_k}{k^{2l}} \sin(kx)$$

and

$$A_0^{-2l+1}\psi(x) = -i \sum_{k=1}^{m} \frac{d_k}{k^{2l-1}} \cos(kx).$$

It then follows that y_{2l+1} are odd functions for all $l = 0, \dots, m-1$, and as ϕ is an even function, we find that $By_{2l+1} = Ay_{2l+1} = y_{2l}$. On the other hand, the equalities $By_{2l} = y_{2l-1}$ for $l = 0, \dots, m$ amount to a non-singular system of $m+1$ linear equations in $m+1$ variables $d_0, d_1, \dots, d_m$,

$$\sum_{k=1}^{m} \frac{d_k}{k^{2l+1}} = f_l, \quad l = 0, 1, \dots, m, \tag{3.7}$$

with $f_0 = -i/(2\pi)$, $f_1 = \dots = f_{m-1} = 0$, and $f_m = -id_0/\sqrt{2\pi}$.

Note that $d_0 \neq 0$ as otherwise the system would be inconsistent, so that y_{2m} does not belong to H_0 and thus the chain cannot be extended further. In view of Lemma 3.4, this is true of any other chain of eigen- and associated vectors for the eigenvalue λ_0. As $a_0 \neq 0$, geometric multiplicity of $\lambda_0 = 0$ is equal to one by Lemma 3.1; therefore, λ_0 is a geometrically simple eigenvalue of B of algebraic multiplicity $2m+1$.

The explicit form of ϕ and ψ yields their Fourier coefficients: $a_n = b_n = 0$ if $|n| > m$, $a_n = \sqrt{2\pi}$ for $|n| \le m$, and, finally, $b_n = \sqrt{2\pi} d_n/2i$ for $n = 1, \dots, m$, $b_n = -b_{-n}$ for $n = -m, \dots, -1$, and $b_0 = 0$. Then the characteristic function,

$$F(z) = \sum_{n=-m}^{m} \frac{\overline{a_n} b_n}{n-z} + 1 = \sqrt{2\pi} \sum_{n=1}^{m} \frac{2nb_n}{n^2 - z^2} + 1 = \frac{2\pi}{i} \sum_{n=1}^{m} \frac{nd_n}{n^2 - z^2} + 1$$

is a rational function of the form $P(z)/Q(z)$ with P and Q polynomials of degree at most $2m$. Therefore, F has at most $2m$ zeros counting with multiplicity. On the other hand, it is straightforward to verify that equations (3.7) amount to the relations

$$F(0) = F'(0) = \dots = F^{(2m-1)}(0) = 0,$$

so that $z = 0$ is a zero of F of multiplicity $2m$. This implies that F has no other zeros. In particular, $F(n) \neq 0$ if $n \neq 0$, and thus $\lambda_n = n$ is an algebraically simple eigenvalue of the operator B whenever $|n| > m$.

To sum up, the operator B has an eigenvalue $\lambda_0 = 0$ of algebraic multiplicity $2m + 1$ and simple eigenvalues λ_n for $|n| > m$. Loosely speaking, the rank-one perturbation shifts the eigenvalues $\lambda_{-m}, \dots, \lambda_{-1}$, $\lambda_1, \dots, \lambda_m$ to λ_0 respectively enlarging the multiplicity of the latter.

4. Spectral localization of the operator B

In this section, we study eigenvalue localization of rank-one perturbations of a given self-adjoint operator. Keeping in mind the most important and interesting applications to the differential operators, in addition to $(A1)$ we assume that

(A2) the eigenvalues of A are separated, i.e.,

$$\inf_{n \in I} |\lambda_{n+1} - \lambda_n| =: d > 0. \tag{4.1}$$

We next localize the spectrum of B by studying its characteristic function

$$F(z) = \sum_{k \in I_1} \frac{\overline{a_k} b_k}{\lambda_k - z} + 1.$$

As the Fourier coefficients a_k and b_k of the functions ϕ and ψ are in $\ell_2(I)$, the sequence $\overline{a_k} b_k$ is summable and, due to the Cauchy–Bunyakowsky–Schwarz inequality, its ℓ_1-norm is bounded by $\|\varphi\|\|\psi\|$.

Lemma 4.1. *The spectrum of B lies in the strip*

$$\Pi := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \le \|\varphi\|\|\psi\|\}.$$

Proof. If $z \notin \Pi$, then $|\lambda_k - z| \ge |\operatorname{Im} z| > \|\varphi\|\|\psi\|$, so that

$$\sum_{k \in I_1} \left| \frac{\overline{a_k} b_k}{\lambda_k - z} \right| < \sum_{k \in I_1} \frac{|\overline{a_k} b_k|}{\|\varphi\|\|\psi\|} < 1$$

and $F(z) \ne 0$. □

Further, for an $\varepsilon > 0$ we denote by $C_n(\varepsilon)$ the open circle

$$C_n(\varepsilon) := \{z \in \mathbb{C} \mid |z - \lambda_n| < \varepsilon\}$$

and set

$$R_{N,\varepsilon} := \left\{ z \in \mathbb{C} \mid |\operatorname{Re} z| \ge N\} \setminus \left(\bigcup_{n \in I} C_n(\varepsilon) \right) \right\}$$

Lemma 4.2. *For every $\varepsilon > 0$ there is $N > 0$ such that $R_{N,\varepsilon}$ belongs to the resolvent set of the operator B.*

Proof. For an $\varepsilon > 0$, we choose $N' \in \mathbb{N}$ so that[2]

$$\sum_{|k|\ge N'}^{(1)} |\overline{a_k} b_k| \le \frac{\varepsilon}{4};$$

then, for z outside every circle $C_n(\varepsilon)$,

$$\Big|\sum_{|k|\ge N'}^{(1)} \frac{\overline{a_k} b_k}{\lambda_k - z}\Big| \le \frac{1}{\varepsilon} \sum_{k\in I_1, |k|\ge N'} |a_k b_k| \le \frac{1}{4}.$$

We now take $N'' \in \mathbb{N}$ such that $N'' \ge N' + 4\|\varphi\|\|\psi\|/d$ and choose $N \in \mathbb{N}$ such that $N \ge |\lambda_{N''}|$ and $N \ge |\lambda_{-N''}|$ if $-N'' \in I$. Due to Assumption $(A2)$ it holds that $|\lambda_k - \lambda_m| \ge d|k-m|$; therefore, $|\lambda_k - z| \ge d(N''-N') \ge 4\|\varphi\|\|\psi\|$ whenever $z \in R_{N,\varepsilon}$ and $|k| \le N'$, so that

$$\Big|\sum_{|k|< N'}^{(1)} \frac{\overline{a_k} b_k}{\lambda_k - z}\Big| \le \frac{1}{4}$$

for such z. As a result, for all $z \in R_{N,\varepsilon}$ it holds

$$|F(z)| \ge 1 - \Big|\sum_{k\in I_1} \frac{\overline{a_k} b_k}{\lambda_k - z}\Big| \ge \frac{1}{2};$$

by Lemma 2.3 the set $R_{N,\varepsilon}$ is in the resolvent set of B, and the proof is complete. □

Combining the above two lemmata, we conclude that the spectrum of B is localized in the circles $C_n(\varepsilon)$ and in the rectangular domain

$$\{z \in \mathbb{C} \mid |\operatorname{Re}| \le N,\ |\operatorname{Im} z| \le \|\varphi\|\|\psi\|\},$$

with $N = N(\varepsilon)$ from Lemma 4.2.

Lemma 4.3. *For every $\varepsilon > 0$ there is $K = K(\varepsilon)$ such that for each $n \in I$ with $|n| > K(\varepsilon)$ the circle $C_n(\varepsilon)$ contains precisely one eigenvalue of B.*

Proof. By Lemma 4.2, for all n with large enough $|n|$, the boundary $\partial C_n(\varepsilon)$ of $C_n(\varepsilon)$ is in the resolvent set of B. We next show that the Riesz spectral projections for A and B corresponding to $C_n(\varepsilon)$ are of the same rank (and thus of rank 1) for large enough $|n|$.

For every n with $\partial C_n(\varepsilon) \subset \rho(B)$, we denote by P_n and P'_n the Riesz spectral projectors for A and B respectively on the root subspaces

[2]Throughout this section, the symbol $\sum^{(1)}$ denotes summation over the index set I_1

corresponding to the eigenvalues inside $C_n(\varepsilon)$,

$$P_n = \frac{1}{2\pi i}\int_{C_n(\varepsilon)} (A-z)^{-1}\,dz, \qquad P_n' = \frac{1}{2\pi i}\int_{C_n(\varepsilon)} (B-z)^{-1}\,dz.$$

By the Krein resolvent formula (2.3), we get

$$P_n - P_n' = \frac{1}{2\pi i}\int_{C_n(\varepsilon)} \frac{dz}{F(z)}\langle\,\cdot\,,(A-\overline{z})^{-1}\varphi\rangle(A-z)^{-1}\psi.$$

As the norm of a rank-one operator $\langle\,\cdot\,u\rangle v$ is equal to $\|u\|\|v\|$ and, as proved in Lemma 4.2, $|F(z)|\ge 1/2$ on $\partial C_n(\varepsilon)$ for large enough $|n|$, we conclude that

$$\|P_n - P_n'\| \le 2\varepsilon \max_{z\in C_n(\varepsilon)} \|(A-\overline{z})^{-1}\varphi\|\|(A-z)^{-1}\psi\|$$

for such n. Observe now that for every vector $u=\sum c_k v_k$ we have

$$\|(A-z)^{-1}u\|^2 = \sum_{k\in I}\frac{|c_k|^2}{|\lambda_k - z|^2};$$

applying the Lebesgue dominated convergence theorem, we conclude that

$$\max_{z\in C_n(\varepsilon)} \|(A-z)^{-1}u\|^2 \to 0$$

as $|n|\to\infty$. Therefore, $\|P_n - P_n'\|\to 0$ as $|n|\to\infty$; as a result [20, §IV.2], the ranks of the Riesz projectors P_n and P_n' coincide for all n with large enough $|n|$, and the proof is complete. □

Therefore, the operator B has at most finitely many nonsimple eigenvalues; we next prove that there are no other restrictions on them.

Lemma 4.4. *Fix an arbitrary $n\in\mathbb{N}$, an arbitrary sequence $z_1, z_2, \dots, z_n$ of pairwise distinct complex numbers, and an arbitrary sequence m_1, m_2, …, m_n of natural numbers. Then there is a rank-one perturbation B of the operator A such that, for every $j=1,2,\dots,n$, the number z_j is an eigenvalue of B of algebraic multiplicity m_j.*

Proof. For simplicity, we assume that none of z_j is in the spectrum of A; the changes to be made otherwise are not very significant, cf. Lemma 3.7 and Example 3.8.

Set $N := m_1 + m_2 + \cdots + m_n$; we will construct a rank-one perturbation B of A with

$$\varphi = \sum_{k=1}^{N} a_k v_k, \qquad \psi = \sum_{k=1}^{N} b_k v_k.$$

According to Lemma 2.3, it suffices to choose a_k and b_k in such a way that the characteristic function F of (3.1) has zeros $z_1, z_2, \dots, z_n$ of multiplicity $m_1, m_2, \dots, m_n$ respectively. Set $c_k := \overline{a_k} b_k$, $k = 1, 2, \dots, n$; then

$$F(z) = \sum_{k=1}^{n} \frac{c_k}{\lambda_k - z} + 1,$$

and the equalities $F(z_k) = F'(z_k) = \dots = F^{(m_k-1)}(z_k) = 0$ lead to an inhomogeneous system of N equations in the variables $c_1, c_2, \dots, c_N$:

$$\sum_{k=1}^{N} \frac{c_k}{(\lambda_k - z_j)^m} + \delta_{m1} = 0, \quad j = 1, 2, \dots, n, \quad m = 1, 2, \dots, m_j, \tag{4.2}$$

with δ_{m1} being the Kronecker delta. By Lemma 4.5 below, the coefficient matrix of the above system is non-singular; therefore, the system possesses a unique solution $c_1, c_2, \dots, c_N$. It remains to take $a_k = 1$ and $b_k = c_k$ for $k = 1, 2, \dots, N$, and the proof is complete. □

Lemma 4.5. *The coefficient matrix of system* (4.2) *is non-singular.*

Proof. For pairwise distinct numbers $\omega_1, \omega_2, \dots, \omega_N$ from the resolvent set of A, we introduce the $N \times N$ Cauchy matrix M with entries

$$(M)_{jk} = \frac{1}{\lambda_k - \omega_j}, \qquad j, k = 1, \dots, N.$$

It is non-singular and has determinant equal to

$$D(\omega_1, \omega_2, \dots, \omega_N) = \frac{\prod\prod_{j>k}(\lambda_j - \lambda_k)(\omega_j - \omega_k)}{\prod_j \prod_k (\lambda_j - \omega_k)}. \tag{4.3}$$

We set $C := \prod\prod_{j>k}(\lambda_j - \lambda_k)$ for brevity.

Taking the derivative of that determinant in ω_2 and setting $\omega_2 = \omega_1 = z_1$, we get the determinant of the matrix M_2, whose first and second rows have entries

$$\frac{1}{\lambda_k - z_1} \quad \text{and} \quad \frac{1}{(\lambda_k - z_1)^2}, \qquad k = 1, 2, \dots, N,$$

respectively, and the other rows are as in the matrix M. By (4.3), we have

$$D(\omega_1, \omega_2, \dots, \omega_N) = (\omega_2 - \omega_1) D_2(\omega_1, \omega_2, \dots, \omega_N),$$

so that

$$\frac{\partial}{\partial \omega_2} D(z_1, \omega_2, \dots, \omega_N)\Big|_{\omega_2 = z_1} = D_2(z_1, z_1, \omega_3, \dots, \omega_N).$$

Explicit calculations give

$$\begin{aligned}\det M_2 &= D_2(z_1, z_1, \omega_3, \dots, \omega_N)\\ &= C\prod_{j>2}(\omega_j - z_1)^2 \frac{\prod\prod_{j>k>2}(\omega_j - \omega_k)}{\prod_j(\lambda_j - z_1)^2\prod_{k>2}(\lambda_j - \omega_k)} \neq 0.\end{aligned}$$

Next, we take the second derivative of $D_2(z_1, z_1, \omega_3, \dots, \omega_N)$ in ω_3 and set $\omega_3 = z_1$; this becomes the determinant $D_3(z_1, z_1, z_1, \omega_4, \dots, \omega_N)$ of the matrix M_3 that is M_2 with its third row replaced by

$$\frac{2}{(\lambda_k - z_1)^3}, \qquad k = 1, 2, \dots, N.$$

On the other hand,

$$\begin{aligned}\det M_3 = D_3(z_1, z_1, z_1, \omega_4, \dots, \omega_N) &= \frac{\partial^2}{\partial\omega_3^2} D_2(z_1, z_1, \omega_3, \dots, \omega_N)\Big|_{\omega_3 = z_1}\\ &= 2C\prod_{j>3}(\omega_j - z_1)^3 \frac{\prod\prod_{j>k>3}(\omega_j - \omega_k)}{\prod_j(\lambda_j - z_1)^3\prod_{k>3}(\lambda_j - \omega_k)} \neq 0.\end{aligned}$$

On each next step, we repeat a similar procedure with the next row and variable until we reach row number m_1.

After that, we set $\omega_{m_1+1} = z_2$, take the derivative in ω_{m_1+2} at $\omega_{m_1+2} = z_2$, and repeat with the subsequent rows until we reach row number $m_1 + m_2$. Clearly, the operations described above can be performed on separate groups of variables ω_l with $l = m_1 + \dots + m_j + 1, m_1 + \dots + m_j + 2, \dots, m_1 + m_2 + \dots + m_{j+1}$ independently. At the end, the determinant of the coefficient matrix of the system (4.2) is found explicitly to be

$$\frac{\prod_{k=1}^N\prod_{j=k+1}^N(\lambda_j - \lambda_k)\prod_{k=1}^n\prod_{j=k+1}^n(z_j - z_k)^{m_j+m_k}}{\prod_{j=1}^N\prod_{k=1}^n(\lambda_j - z_k)^{m_j}} \neq 0,$$

and the proof is complete. □

Remark 4.6. *In the paper [14], it is proved that the operators A and B have the same number of eigenvalues in special increasing rectangles exhausting the whole complex plane $\mathbb{C}$. Combined with the results of Lemmata 4.2 and 4.3, this allows an enumeration of the eigenvalues of B as μ_n, $n \in I$, such that each value μ_n is repeated according to its multiplicity and $\mu_n - \lambda_n \to 0$ as $|n| \to \infty$.*

We summarize the above results in the following theorem.

Theorem 4.7. *Assume that A is an operator in a Hilbert space H satisfying assumptions* (A1) *and* (A2) *and B is its rank-one perturbation* (2.1). *Then*

(i) *all eigenvalues of B of sufficiently large absolute value are localized within ε-neighbourhood of the eigenvalues of A and thus are simple;*
(ii) *the eigenvalues of B can be enumerated as μ_n, $n \in I$, so that $\mu_n - \lambda_n \to 0$ as $|n| \to \infty$;*
(iii) *geometric multiplicity of every eigenvalue of B is at most 2, and multiplicity 2 is only possible when the corresponding eigenspace of A is reducing for B.*

Moreover, for every prescribed finite set $z_1, z_2, \dots, z_n$ of pairwise distinct complex numbers, and an arbitrary sequence m_1, m_2, $\dots$, m_n of natural numbers there exists a B such that each z_j, $j = 1, 2, \dots, n$, is an eigenvalue of B of algebraic multiplicity m_j.

5. Finite-dimensional case

The analysis of Section 3 allows to essentially complement the results in the finite-dimensional case. Namely, assume that A is a Hermitian matrix in $\mathbb{C}^n$ with pairwise distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and normalized (column) eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and define the *generic set* $\mathcal{G}(A)$ of A as

$$\mathcal{G}(A) = \{\mathbf{x} \in \mathbb{C}^n \mid \langle \mathbf{x}, \mathbf{v}_k \rangle_{\mathbb{C}^n} \neq 0, \quad k = 1, 2, \dots, n\}.$$

Then we have the following generalization of the result of [21].

Theorem 5.1. *Under the above assumptions, let $\boldsymbol{\varphi}$ be a vector from the generic set $\mathcal{G}(A)$. Then for any natural number k, any pairwise distinct complex numbers $z_1, z_2, \dots, z_k$, and any natural numbers $m_1, m_2, \dots, m_k$ satisfying $m_1 + m_2 + \dots + m_k = n$, there is a unique vector $\boldsymbol{\psi} \in \mathbb{C}^n$ such that the rank-one perturbation $B = A + \boldsymbol{\psi}\boldsymbol{\varphi}^\top$ of the matrix A has eigenvalues $z_1, z_2, \dots, z_k$ of corresponding multiplicities $m_1, m_2, \dots, m_k$.*

Similarly, for every fixed $\boldsymbol{\psi} \in \mathcal{G}(A)$ there is a unique $\boldsymbol{\varphi} \in \mathbb{C}^n$ such that B has the eigenvalues z_j of prescribed multiplicities m_j, $j = 1, 2, \dots, k$.

Proof. Denote by $\sigma_0(A)$ the common part of the spectrum $\sigma(A)$ of A and the set $\{z_1, z_2, \dots, z_k\}$, by $\sigma_1(A) := \sigma(A) \setminus \sigma_0(A)$ the remaining part of $\sigma(A)$, and let $I_\ell := \{j \mid \lambda_j \in \sigma_p(A)\}$, $\ell = 0, 1$, be the corresponding index sets. We update the multiplicities m_j to m_j' with

$$m_j' := \begin{cases} m_j - 1, & z_j \in \sigma(A); \\ m_j, & z_j \notin \sigma(A); \end{cases} \tag{5.1}$$

and set

$$F(z) := \frac{\prod_{j=1}^{k}(z-z_j)^{m'_j}}{\prod_{j\in I_1}(z-\lambda_j)}. \tag{5.2}$$

Denoting by $-c_j$ the residue of the function F at the point $z = \lambda_j$, $j \in I_1$, we conclude that F can be written in the form

$$F(z) = \sum_{j\in I_1} \frac{c_j}{\lambda_j - z} + 1. \tag{5.3}$$

Denote by $a_j = \langle \boldsymbol{\varphi}, \mathbf{v}_j\rangle_{\mathbb{C}^n}$, $j = 1, 2, \dots, n$, the coefficients of the vector $\boldsymbol{\varphi}$ in the basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. By assumption, no a_j vanishes, and we set $b_j := c_j/\overline{a_j}$ for $j \in I_1$ and $b_j = 0$ for $j \in I_0$, and define the vector ψ via

$$\boldsymbol{\psi} = \sum_{j=1}^{n} b_j \mathbf{v}_j = \sum_{j\in I_1} b_j \mathbf{v}_j.$$

It follows from the results of Section 3 that the characteristic function of the matrix $B = A + \boldsymbol{\psi}\boldsymbol{\varphi}^\top$ coincides with the above function F; therefore, the matrix B has eigenvalues $z_1, z_2, \dots, z_k$ and the multiplicity of the eigenvalue z_j is m'_j if $z_j \notin \sigma(A)$ or $m'_j + 1$ otherwise.

The second part is proved in a similar manner, by interchanging the rôles of a_n and b_n. □

If the vector $\boldsymbol{\varphi}$ is not in the generic set $\mathcal{G}(A)$ of A, the above theorem has the following analogue.

Theorem 5.2. *Under the above assumptions on the matrix A, take a nonzero vector $\boldsymbol{\varphi} = \sum_{j=1}^{n} a_j \mathbf{v}_j \in \mathbb{C}^n$ and set $I_0 := \{j \mid a_j = 0\}$ and $\sigma_0(A) := \{\lambda_j \mid j \in I_0\}$. Then for every natural k, every set $S = \{z_1, z_2, \dots, z_k\}$ of k pairwise distinct complex numbers obeying $S \cap \sigma(A) = \sigma_0(A)$, and every sequence $m_1, m_2, \dots, m_k$ of natural numbers with $m_1 + m_2 + \dots + m_k = n$ there is a vector $\boldsymbol{\psi} \in \mathbb{C}^n$ such that the matrix $B = A + \boldsymbol{\psi}\boldsymbol{\varphi}^\top$ has eigenvalues $z_1, z_2, \dots, z_k$ of multiplicities $m_1, m_2, \dots, m_k$ respectively.*

A similar statement holds with the rôles of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ interchanged.

Proof. The fact that the set $\sigma_0(A)$ must be in the spectrum of B is proved in Lemma 2.3. We denote by $\sigma_1(A)$ the spectrum of A not in $\sigma_0(A)$ and set I_1 to be the corresponding set of indices. Reducing by 1 the multiplicity of each z_j from S and denoting the resulting multiplicities by m'_j as in (5.1), we construct the function F of (5.2) and observe that it assumes the form (5.3), with uniquely determined residues $-c_j$, $j \in I_1$. Then we define b_j for such j from the relation $\overline{a}_j b_j = c_j$, and fix arbitrarily b_j for $j \in I_0$.

By Lemmata 3.5 and 3.7, the numbers z_j not in $\sigma_0(A)$ are eigenvalues of the matrix B of multiplicity m'_j, while those in $\sigma_0(A)$ have multiplicity $m'_j + 1$. The proof is complete. □

Remark 5.3. *We can conclude from the above proof that the coordinates of the vector* $\boldsymbol{\psi}$ *in the basis* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ *for* $j \in I_0$ *are not fixed; therefore, there is an* $|I_0|$*-dimensional affine set of such vectors producing the required spectrum.*

6. Concluding remarks

It should be noted that some restrictions imposed on A can be relaxed. For instance, self-adjointness of A is not essential; the proof with minor amendments will work for rank-one perturbations of every normal operator with simple discrete spectrum, or even in the case when the eigenvectors of A can be chosen to form a Riesz basis of H. Simplicity of the eigenvalues of A can also be dropped; however, this will result in a more complicated Jordan structure of the root subspaces of B, cf. [12]. Also, the operator A may possess, in addition to an infinite discrete spectrum, a non-trivial essential component; the results we proved have natural generalization to this case as well.

Finally, this study has found its continuation in [14], in which a complete characterization of all possible spectra of rank-one perturbations (2.1) of self-adjoint operators A with simple discrete spectrum is given.

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